

Quantization of generic chaotic 3D billiard with smooth boundary II: structure of high-lying eigenstates

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Abstract This is the first survey of highly excited eigenstates of a chaotic 3D billiard. We introduce a strongly chaotic 3D billiard with a smooth boundary and we manage to calculate accurate eigenstates with sequential number (of a 48-fold desymmetrized billiard) about 45,000. Besides the brute-force calculation of 3D wavefunctions we propose and illustrate another two representations of eigenstates of quantum 3D billiards: (i) normal derivative of a wavefunction over the boundary surface, and (ii) ray — angular momentum representation. The majority of eigenstates is found to be more or less uniformly extended over the entire energy surface, as expected, but there is also a fraction of strongly localized — scarred eigenstates which are localized either (i) on to classical periodic orbits or (ii) on to planes which carry (2+2)-dim classically invariant manifolds, although the classical dynamics is strongly chaotic and non-diffusive.

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The structure of individual quantum eigenstates of classically chaotic closed systems is one of the most important and difficult questions of quantum chaos. Understanding of localization properties of highly excited eigenstates can explain almost any statistical property of a quantum system, such as e.g. statistics of energy levels (see e.g. [1]) or statistical distribution of transition amplitudes [2]. For classically fully chaotic — ergodic billiards we have a theorem [3] which states that asymptotically, in the semiclassical limit $\hbar \rightarrow 0$, probability density of eigenstates should become uniformly delocalized — extended over the billiard domain, i.e. phase space distributions of eigenstates should approach a uniform distribution over the energy surface. The fluctuations of an eigenfunction $\Psi(\vec{r})$ are also expected to obey a universal law for classically fully chaotic systems [4]. For systems having a time-reversal symmetry, $\Psi(\vec{r})$ is expected to behave like a real Gaussian random variable Ψ with a uniform probability distribution $d\mathcal{P}/d\Psi = 1/(\sqrt{2\pi}\sigma) \exp(-\Psi^2/2\sigma^2)$ where inverse variance $1/\sigma^2$ is constant and equal to the volume of the billiard.

For finite values of Planck's constant \hbar there may be considerable deviations from this semiclassical limit: (i) There may be classical adiabatic invariants, which change slowly and diffusively in the course of time, or partial barriers in phase space — cantori, so that classical orbits spend a long time to uniformly fill entire energy surface. If this time is longer than Heisenberg (*break*) time $t_{break} = \hbar d(E)$, where $d(E)$ is the density of states, then one should expect strong localization of quantum eigenstates on such approximately disconnected components of energy surface [1]. (ii) But even if classical dynamics quickly explores the entire energy surface w.r.t. t_{break} one can get strong enhancements of probability density of eigenstates on the least unstable classical periodic orbits — the so-called *scars* [5].

Semiclassical theory has been developed, which explains the effect of classical periodic orbits onto the composition of eigenstates within an energy interval much larger than the mean level spacing [6], but there is no adequate theory to predict the structure of individual eigenstates. One can use a heuristic argument due to Heller [5], saying that an eigenstate is scarred by a given classical periodic orbit if a Gaussian wavepacket, which is launched along the orbit, interferes constructively with itself after one period. But in chaotic systems there are many periodic orbits and it is typically impossible to tell afortiori which orbits will scar most intensively. (The rule that these are the least unstable ones holds only on average.)

In the literature there exists a vast amount of numerical evidence on scarring in chaotic systems with two freedoms (see e.g. [7],[8],[9]), while the structure of comparably high-lying eigenstates of three and higher dimensional chaotic systems have numerically not yet been studied, so it is not known whether classical periodic orbits can scar eigenstates in higher dimensions or not. Heuristic geometrical argument might suggest that scarring by individual periodic orbit is less effective in 3D than in 2D since more phase space is available for a wavepacket to

disperse before it can interfere with itself. But scarring by families of similar periodic orbits [7] may be effective in higher dimensions. Actually, as approaching the limit $\hbar \rightarrow 0$, quantum invariant states — eigenstates should approach classical invariant states which are spanned by characteristic functions on the classically invariant (sub)manifolds in phase space; but only the *microcanonical* distribution is stable for long times in a fully chaotic system, so it is the only one which survives the semiclassical limit. Periodic orbits are only special cases of classically invariant submanifolds in the phase space of a dynamical system ((1+1)D invariant submanifolds when they are parametrized by the energy E). In a 3D system there may exist also (2+2)D invariant submanifolds in 6D phase space which can have 2D (or 3D) projection onto 3D configuration space supporting a 2D (or 3D) scar. This type of localization is a genuine 3D effect which is yet to be numerically confirmed.

The purpose of this paper is to give the first survey of the high-lying eigenfunctions of a strongly chaotic 3D system. We have defined a family of *generic* 3D billiards with a smooth C^∞ boundary (see also [11]). Since no *generic* system is known to be rigorously ergodic we have defined a two-parameter family of 3D billiards whose shapes are given by simplest smooth deformations of a sphere: The radial distance $r_B(\vec{n})$ from the origin to the boundary as a function of the direction $\vec{n}, n^2 = 1$, is

$$r_B(\vec{n}) = 1 + a(n_x^4 + n_y^4 + n_z^4) + b n_x^2 n_y^2 n_z^2, \quad (1)$$

and contains the two lowest order terms which preserve the cubic symmetry (the first two fully symmetric type (α) cubic harmonics after Von Lage and Bethe [10]). After a careful numerical exploration of a parameter space (a, b) we have decided to chose: $a = -1/5, b = -12/5$. For these values of the parameters the classical billiard is strongly chaotic: nondiffusive, without partial barriers in phase space, and with large average maximal liapunov exponent $\langle \lambda_{max} \rangle = 0.54$ meaning that roughly after five bounces we loose one digit of information of a classical orbit. There is a tiny regular component of phase space whose relative volume has been estimated to be $\rho_1 \approx 10^{-3}$. The billiard domain is marginally convex and there are few isolated neutrally stable — parabolic periodic orbits which touch the boundary at the points of zero curvature radii.

The billiard is invariant under 48-fold cubic symmetry group O_h . So we desymmetrize it: In the following we consider a 1/48 of an original billiard (1) with $x \geq y \geq z \geq 0$, i.e. we consider a 3D billiard in a domain, bounded by the boundary surface $\mathcal{B} = \{\vec{r}; r_B(\vec{r}/r) = r\}$, and the symmetry planes $z = 0$, $x = y$, and $y = z$. The symmetry planes in phase space, $\mathcal{R}_z = \{z = 0, p_z = 0\}$, $\mathcal{Q}_{xy} = \{x = y, p_x = p_y\}$, and $\mathcal{Q}_{yz} = \{y = z, p_y = p_z\}$, and the boundary surface with the tangential momentum $\mathcal{S} = \{r_B(\vec{r}/r) = r, \vec{p} \cdot \nabla(r_B(\vec{r}/r) - r) = 0\}$ are invariant

(2+2)D submanifolds with respect to the classical dynamics which have 2D projections onto 3D configuration space. When $b = 12a$, as is the case with our choice, the billiard (1) possesses another less trivial example of 4D invariant submanifold, namely $\mathcal{T} = \{x = y + z, p_x = p_y + p_z\}$, with a 2D configurational projection, the plane $x = y + z$.

Quantum eigenfunctions of our billiard (1) carry irreducible representations of a cubic group. We have decided to study only the singlet eigenstates belonging to fully symmetric 1D irrep. This corresponds to a study of a desymmetrized billiard, Helmholtz equation $(\nabla^2 + k^2)\Psi_k(\vec{r}) = 0$, with Von Neuman boundary conditions on the symmetry planes $z = 0, x = y$ and $y = z$, and Dirichlet boundary conditions on the boundary surface \mathcal{B} . Quantum eigenfunctions $\Psi(\vec{r})$ have been calculated using 3D generalization of *scaling method*, proposed recently by Vergini and Saraceno [12]. For billiards whose shape is geometrically not very far from a sphere, we propose to use spherical waves $\phi_{klm}(\vec{r}) = j_l(kr)Y_{lm}(\vec{r}/r)$ instead of plane waves $\exp(i\vec{k} \cdot \vec{r})$ as the basis of scaling functions which already solve the Helmholtz equation but do not satisfy the boundary conditions. We define the efficiency of the basis of spherical scaling functions [11] as $\eta = V/V_{sph}$ where V is the volume of the billiard, and V_{sph} is the volume of the smallest sphere enscribed to the billiard which has radius $r_{max} = \max\{r_B(\vec{r}/r)\}$. For a desymmetrized billiard one should use only the linear combinations of spherical waves which already satisfy Von Neuman boundary conditions on the symmetry planes. These are the products of spherical Bessel functions and the fully symmetric cubic harmonics (of type (α) after Von Lage and Bethe [10]) — *cubic waves*, labeled by α ,

$$\phi_{k\alpha}(\vec{r}) = j_{l_\alpha}(kr) \sum_{j=0}^{l_\alpha/4} c_{\alpha j} (Y_{l_\alpha, 4j}(\vec{r}/r) + Y_{l_\alpha, -4j}(\vec{r}/r)).$$

The coefficients $c_{\alpha j}$ are determined by the Gram-Schmidt orthogonalization of the columns of the projector onto the fully symmetric irrep of the cubic group $O_h, P = (1/48) \sum_{G \in O_h} G$, in the basis of spherical harmonics $Y_{l_\alpha, 4j}$ with even angular momentum $l_\alpha = 2n$, $P_{jj'}^{(n)} = (1 + 2D^{(2n)}(0, \pi/2, 0)_{4j, 4j'})/3$ (using the standard notation [13]).

For an accurate quantization at fixed wavenumber k we need cubic waves α with angular momentum l_α up to

$$l_{max}(k) := kr_{max} + \Delta l_{evanescent}.$$

The number of included evanescent angular momenta $\Delta l_{evanescent}$ is found to be proportional to the number of accurate digits of the results, and being much smaller than kr_{max} . For any *even* l , $l \leq l_{max}$ there are $\lfloor l/12 \rfloor + 1 - \delta_{r,2}$ fully symmetric cubic waves, where $l = 12\lfloor l/12 \rfloor + r, 0 \leq$

$r < 12$, which sum up to the total number

$$N_{CW}(k) = l_{max}(k)^2/48 + l_{max}(k)/4 + \mathcal{O}(1)$$

of fully symmetric cubic waves $\phi_{k\alpha}(\vec{r})$ which should accurately capture the eigenfunction $\Psi_k(\vec{r})$ at the wavenumber k

$$\Psi_k(\vec{r}) = \sum_{\alpha} \psi_{\alpha} \phi_{k\alpha}(\vec{r}). \quad (2)$$

One should find such k and coefficients ψ_{α} that $\Psi(\vec{r})$ satisfy the boundary condition on the boundary surface, $\Psi|_{\mathcal{B}} = 0$. Following [12] we minimize a special *boundary norm*

$$f(k) = \int_{\mathcal{B}} \frac{d^2 S}{\vec{\nu} \cdot \vec{r}} |\Psi_k(\vec{r})|^2, \quad (3)$$

where $d^2 S$ is a boundary surface element and $\vec{\nu} = \nabla(r_B(\vec{r}/r) - r)/|\dots|$ is a unit vector normal to the boundary \mathcal{B} , by solving the following generalized eigenvalue problem

$$\sum_{\alpha'} \frac{d}{dk} F_{\alpha\alpha'}(k_0) \psi_{\alpha'} = \lambda \sum_{\alpha'} F_{\alpha\alpha'}(k_0) \psi_{\alpha'}, \quad (4)$$

where

$$F_{\alpha\alpha'}(k_0) = \int_{\mathcal{B}} \frac{d^2 S}{\vec{\nu} \cdot \vec{r}} \phi_{k_0\alpha}(\vec{r}) \phi_{k_0\alpha'}(\vec{r}),$$

is a positive definite matrix $N_{CW}(k_0) \times N_{CW}(k_0)$. Every matrix element is an integral over the surface of the a sphere (or over 1/48 of it), with integration measure written in spherical coordinates (θ, ϕ)

$$\frac{d^2 S}{\vec{\nu} \cdot \vec{r}} = d \cos \theta d\phi \sqrt{r_B^2 + (\partial_{\theta} r_B)^2 + (\partial_{\phi} r_B / \sin \theta)^2},$$

where $r_B = r_B(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$, which is accurately numerically evaluated using a product grid of a Gaussian quadrature for $\cos \theta$ integration $\int_0^1 d \cos \theta$ with $l_{max}/2$ (positive) nodes and a uniform grid for ϕ integration $\int_0^{\pi/4} d\phi$ with $l_{max}/4$ nodes.² The eigenvalues $\lambda^{(n)}$ of largest modulus are good numerical estimates of billiard's eigenvalues $k^{(n)} \approx k_0 - 2/\lambda^{(n)}$. Due to the scaling symmetry, $\phi_{k,\alpha}(\vec{r}) = \phi_{k_0,\alpha}(k\vec{r}/k_0)$, the corresponding eigenvectors $\psi_{\alpha}^{(n)}$ yield very accurate eigenfunctions $\Psi^{(n)}(\vec{r})$, via expansion (2), with error being of the order $\mathcal{O}((k^{(n)} - k_0)^4)$.

² We have also implemented an optimal cubature formula due to Lebedev for accurate numerical integration over the sphere, which reduces the size of a grid for integration over the desymmetrized boundary surface \mathcal{B} , w.r.t. product grid, by a factor of 9/2, and the total numerical labour by 20 – 40%. Unfortunately we found that numerical computation of the grid points and weights of such an optimal grid for large l_{max} turns out to be extremely difficult numerically, more than the problem itself.

Accurate eigenvalues have been obtained by minimizing the boundary norm $f(k)$ (as expanded in a Taylor series around k_0 up to 8th order) for fixed coefficients $\psi_\alpha^{(n)}$. The number of converged eigenvalues $k^{(n)}$ or eigenvectors $\psi_\alpha^{(n)}$ in a single diagonalization of (4) depends on the required accuracy, empirically we find [11] that it is proportional to k^2 .

We have chosen $k_0 = 360.0$ and calculated a stretch of 64 consecutive eigenstates with eigenvalues $k^{(n)}$ from a narrow window centered around k_0 , corresponding to sequential quantum number of a desymmetrized billiard $\mathcal{N}(k_0) = 45,103 \pm 2$ (using 3D Weyl formula [11]). Computation has been performed with $l_{\text{evanescent}} = 40$ (and $l_{\text{max}} = 364$) and the boundary norm (3) of normalized wavefunctions $\Psi^{(n)}$ has ranged between $5 \cdot 10^{-9}$ and $5 \cdot 10^{-8}$. However, it is much easier to calculate high-lying eigenfunctions of a 3D billiard, than to analyze and faithfully represent them without loosing much of information (since we can plot functions of at most two variables). For the presentation we have picked a sample of 12 eigenstates out of 64: 11 consecutive states and a specially chosen state with the criterion of being most intensively localized.

The first representation: *normal derivative on the boundary surface* $\vec{\nu} \cdot \nabla \Psi^{(n)}|_{\mathcal{B}}$. Although this representation formally contains all the information about a given eigenstate, it explicitly displays only a very limited portion of configuration space and it is not very suitable to detect localization which can take place strictly inside the billiard region. Nevertheless, it is the easiest to be calculated numerically, $\vec{\nu} \cdot \nabla \Psi^{(n)} = \sum_\alpha \psi_\alpha^{(n)} \vec{\nu} \cdot \nabla \phi_{k^{(n)},\alpha}$, one only has to calculate the normal derivatives of the cubic waves on \mathcal{B} . In figure 1 we show the nodal lines of $\vec{\nu} \cdot \nabla \Psi^{(n)}$ on \mathcal{B} and the contour plots of its magnitude. Actually, for easier planar presentation we map the desymmetrized boundary \mathcal{B} , which has a shape of a deformed spherical triangle with angles $\pi/4, \pi/2, \pi/3$, at the points $A_1 = (1, 0, 0)r_B(1, 0, 0)$, $A_2 = 2^{-1/2}(1, 1, 0)r_B(2^{-1/2}, 2^{-1/2}, 0)$, $A_3 = 3^{-1/2}(1, 1, 1)r_B(3^{-1/2}, 3^{-1/2}, 3^{-1/2})$, where, respectively, 4-fold, 2-fold, and 3-fold symmetry axis penetrates \mathcal{B} , onto the half-square $\{(\phi, \chi), 0 \leq \chi \leq \phi \leq \pi/4\}$:

$$\begin{aligned} \vec{r}(\phi, \chi) &= \vec{n}(\phi, \chi)r_B(\vec{n}(\phi, \chi)), \\ \vec{n}(\phi, \chi) &= ((1 - z(\chi)^2)^{-1/2} \cos \phi, (1 - z(\chi)^2)^{-1/2} \sin \phi, z(\chi)), \\ z(\chi) &= (1 + (\sin \chi)^{-2})^{-1/2} \end{aligned}$$

Note that this mapping is not far from being area and angle preserving. So we plot

$$B^{(n)}(\phi, \chi) = \vec{\nu} \cdot \nabla \Psi^{(n)}|_{\vec{r}(\phi, \chi)}.$$

Figure 1 appears similar to the pictures of eigenfunctions of 2D chaotic billiards. On diagrams we indicate the projections of known (2+2)D invariant submanifolds: the symmetry planes are

mapped on the boundary of the triangle $\mathcal{R}_z \rightarrow \chi = 0$, $\mathcal{Q}_{xy} \rightarrow \phi = \pi/4$, $\mathcal{Q}_{yz} \rightarrow \chi = \phi$, while the invariant plane \mathcal{T} is mapped on the curve $\sin \chi = \cos \phi - \sin \phi$. Most of states appear more or less random (the statistical distribution of normal derivatives over \mathcal{B} has been analyzed and found to be very close to a Gaussian for most of states). However the state 12 is strongly enhanced in the vicinity of point A_1 , as well as the states 7, 8, 10, while the states 7, 10 are significantly enhanced in the vicinity of the projected manifold \mathcal{T} .

Uniform representation of 3D eigenfunctions without loosing essential information: *ray — angular momentum representation*. In the momentum — angular momentum representation, quantum states $|\Psi\rangle$ are labeled by eigenvalues of the three commuting observables, \vec{p}^2 , $(\vec{r} \times \vec{p})^2$, and $(\vec{p} \times \vec{r})_z$, namely $\hbar^2 k^2$, $\hbar^2 l(l+1)$, and $\hbar m$,

$$\langle klm|\Psi\rangle = \int_{4\pi} d^2\vec{n} Y_{lm}(\vec{n}) \int_0^{r_B(\vec{n})} dr r^2 j_l(kr) \Psi(\vec{r}).$$

Note that eigenstates in the momentum — angular momentum representation, normalized as $\sum_{lm} \int_0^\infty dk k^2 |\langle klm|\Psi^{(n)}\rangle|^2$, are strongly localized around $k^{(n)}$ in the momentum coordinate k , which is classically a constant of the motion. The structure of eigenfunctions going perpendicularly to the energy shell $k \approx k^{(n)}$ is somehow trivial ($\langle klm|\Psi\rangle \approx 0$, if $|k - k^{(n)}| r_{max} \gg 1$) and certainly less interesting than the structure inside the shell. Therefore we define a ray—angular momentum representation of 3D wavefunctions

$$R_{lm}[\Psi] = \int_0^\infty dk k^2 |\langle klm|\Psi\rangle|^2, \quad (5)$$

which gives the total probability for a particle with arbitrary momentum $\hbar k$ moving along a ray with fixed angular momentum (l, m are the natural coordinates on the spherical energy shell in the momentum space). Numerical calculations with formula (5) are very tedious since they require calculation of the full momentum—angular momentum representation $\langle klm|\Psi\rangle$ first. Using well known relations for spherical harmonics and Bessel functions one can reformulate eq. (5) giving more practical expression

$$R_{lm}[\Psi] = \frac{\pi}{2} \int_0^{r_{max}} dr r^2 \left(\int_{4\pi} d^2\vec{n} \theta(r_B(\vec{n}) - r) Y_{lm}(\vec{n}) \Psi(r\vec{n}) \right)^2$$

For the eigenstates $\Psi^{(n)}$ of our billiard we have $R_{lm}^{(n)} = 0$ unless $l = 2n, m = 4j$ and $R_{l,m}^{(n)} = R_{l,-m}^{(n)}$ (due to the symmetry), and the integration over r can be simplified up to the radius of the largest

inscribed sphere, $r_{min} = \min\{r_B(\vec{r}/r)\}$,

$$R_{l,4j}^{(n)} = \frac{\pi}{2} \left(\sum_{\alpha}^{l_{\alpha}=l} \psi_{\alpha}^{(n)} c_{\alpha j} \right)^2 \int_0^{r_{min}} dr r^2 j_l^2(k^{(n)}r) \\ + \frac{\pi}{2} \int_{r_{min}}^{r_{max}} dr r^2 \left[\int_{4\pi} d^2\vec{n} \theta_{\epsilon}(r_B(\vec{n}) - r) Y_{l,4j}(\vec{n}) \Psi^{(n)}(r\vec{n}) \right]^2.$$

The numerical integrals over r have been performed using another (sufficiently dense) Gaussian quadrature, and smooth (Fermi) approximation for the step function $\theta_{\epsilon}(x) = 1/(1 + \exp(-x/\epsilon))$, where $\epsilon < 2\pi/k^{(n)}$. For chaotic (microcanonical) eigenfunctions, $R_{lm}^{(n)}$ are expected to fluctuate around smooth microcanonical classical angular momentum distribution $R_{lm}^{(cl)} = \langle \delta(\hbar l - |\vec{r} \times \vec{p}|) \delta(\hbar m - (\vec{r} \times \vec{p})_z) \rangle$ on the classically allowed region of lm where $R_{lm}^{(cl)}$ is nonzero. Strong systematic deviations indicate localization or scarring. However, the statistical fluctuations of $\rho_{lm} = R_{lm}^{(n)}/R_{lm}^{(cl)}$ are very large. We have numerical evidence for the *stationarity* and *Lorentzian tails* of the statistical distribution of ρ_{lm} , $d\mathcal{P}/d\rho(\rho \gg 1) = \mathcal{O}(\rho^{-2})$, which has been studied numerically for the full stretch of 64 chaotic eigenfunctions.

In Figure 2 we show the sample of 12 states in the ray—angular momentum representation, which is found to be very *sensitive to localization*, since the coordinates l and $l_z = m$ are the ‘most adiabatic’ observables, (they are the canonical actions for the spherical billiard). Because of strong fluctuations we show (above the diagonals $l = m$ with l and m coordinates interchanged) also the smoothed ray-angular momentum representation

$$\tilde{R}_{lm}^{(n)} = \frac{1}{2\pi\Delta^2} \sum_{l'm'} R_{l'm'}^{(n)} \exp[-((l - l')^2 + (m - m')^2)/(2\Delta^2)],$$

with $\Delta = 3$. Although the system is classically strongly chaotic, having almost GOE spectral statistics [11], we see lots of significantly localized eigenstates. Only the states 1, 2, 3, and 9 are really uniformly extended, the states 7 and 10 are localized on the (2+2)D invariant manifold \mathcal{T} , which is mapped on the line $m = l/\sqrt{3}$, the states 7, 8 and 10 are localized on the symmetric (2+2)D invariant manifolds $\mathcal{R}_{x,y,z}$, which are mapped on the lines $m = 0$ and $m = l$. This representation is convenient also because classical periodic orbits are sets of points since each line segment of an orbit has fixed angular momentum. The state 12 is a strongly enhanced in the neighborhood of an unstable periodic orbit of desymmetrized billiard of length $l = 10.209$ with a pair of unimodular, $\exp(\pm i1.558)$, and a pair of real, 4.000, 0.250, eigenvalues of a monodromy matrix.

Brute-force approach: *3D eigenfunctions* in configuration space. Finally, we have calculated full 3D wavefunctions for the interesting representative eigenstates of our billiard using expansion (2) on a dense mesh of points in 3D configuration space. At every point in space we have then smoothed the probability density $|\Psi|^2$ by averaging over a ball of radius equal to one de Broglie wavelength $2\pi/k$. The regions of high smoothed probability density (being above certain threshold) have been plotted in 3D for states 1, 12, and 8, in figures 3, 4, and 5, respectively. The probability distributions of fluctuation amplitudes $\Psi(\vec{r})$ have been calculated and compared to a Gaussian [4]. The state 1 (figure 3) is shown as an example of a typical chaotic eigenstate which is uniformly extended over the billiard domain with a Gaussian amplitude distribution $d\mathcal{P}/d\Psi$. The state 12 (figure 4) is shown as an example of a 1D scar localized in a neighborhood of an unstable periodic orbit. The state 8 (figure 5) is shown as an example of a 2D scar localized in a neighborhood of an $(2+2)$ D invariant chaotic submanifold \mathcal{R}_z . (Note that the states 7 and 10 have a similar structure but they are associated with two invariant $(2+2)$ D chaotic submanifolds simultaneously, \mathcal{R}_z and \mathcal{T}). Also the amplitude distribution $d\mathcal{P}/d\Psi$ for the localized states 8 and 12 is different from a Gaussian: there is a stronger peak for small amplitudes and slowly decaying tails for large amplitudes.

In a conclusion we wish to emphasize the twofold goal of this paper: (i) the survey of very-high lying eigenstates of a strongly chaotic and generic 3D billiard which has been made possible by (ii) an application of the scaling method for quantization of billiards (which works for the Helmholtz equation in arbitrary dimension), proposed recently [12]. In a preceding paper [11] we found small but significant deviations of the level statistics from the expected ultimate asymptotics (GOE) which have been explained in terms of localization of eigenfunctions. Indeed, besides the majority of extended states with Gaussian random amplitude, we found examples of 1D scars which are strongly enhanced in the vicinity of short unstable classical periodic orbits, and examples of 2D scars which are strongly enhanced in the vicinity of the projections of $(2+2)$ D invariant chaotic submanifolds on to configuration space. This is the first numerical study of the structure of eigenstates of higher dimensional ($D > 2$) chaotic systems in the semi-classical regime (i.e. very large sequential quantum number or small effective Planck's constant). Although the phase space is dynamically more 'connected' than in 2D systems, and the classical dynamics is really uniformly ergodic on a scale of t_{break} , we have found very rich behavior ranging from Gaussian pseudo-random extended states to various types of localized states which are associated with classically invariant structures. This is the result which should motivate further theoretical and numerical study of quantum chaos in higher dimensional systems.

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Figure captions

Figure 1: We show the normal derivative of the wavefunctions for the sample of 12 eigenstates on the mapped boundary surface, the triangle $0 \leq \chi \leq \phi < \pi/4$ (see text). Below the diagonals (abscissa ϕ , ordinate χ) we show the magnitude of the normal derivative (using 10 levels of greyness which change when the square of the function changes for a factor 2.3), and above the diagonals (abscissa χ , ordinate ϕ) we show the nodal lines of the normal derivative. The full curves going through the middle of triangles are the images of (2+2)D invariant submanifold \mathcal{T} , whereas the images of symmetric (2+2)D invariant submanifolds are the boundary lattices $\chi = 0$, $\chi = \phi$, and $\phi = \pi/4$.

Figure 2: Same as in figure 2 but for the ray—angular momentum representation of eigenstates $R_{lm}^{(n)}$ below the diagonals (abscissa $0 \leq l \leq k^{(n)}r_{max}$, ordinate $0 \leq m \leq m$) and their smoothed counterparts $\tilde{R}_{lm}^{(n)}$ above the diagonals (abscissa m , ordinate l). The lines $m = l/\sqrt{3}$ are the images of the relevant (2+2)D invariant manifold \mathcal{T} and the image of the relevant periodic orbit for the state 12 is indicated with small circles. (See text.)

Figure 3: The smoothed 3D probability density of a chaotic eigenfunction (state 1 on figs. 1 and 2). The desymmetrized billiard (we plot the curves along its six edges connecting the four corners 0 (below), A_1 (left), A_2 (middle), and A_3 (right) as small circles) is sliced with horizontal planes $v = (y - z)/\sqrt{2} = n\Delta v$, for $\Delta v = 0.005$ and $n = 0, 1 \dots 90$, into triangular sections. Going from the bottom $n = 0$ to the top $n = 90$, the regions where the smoothed probability density exceeds a threshold, $\Psi_{thr}^2 = 1.1/V$, are plotted and filled with the level of greyness which is proportional to the vertical coordinate v (dark is bottom and bright is top — see grey scale on the right). Hopefully, one can thus get an impression of the 3D structure of eigenfunctions. The inset on the lower right corner of the picture shows the histogram of the amplitude distribution $d\mathcal{P}/d\Psi$ which is in this case in agreement with a Gaussian (dotted curve; vertical dotted lines indicate the positions of ± 1 standard deviation).

Figure 4: The same as in figure 3 but for a strongly localized — scarred eigenfunction (state 12 on figs. 1 and 2). The state is strongly enhanced in the vicinity of an unstable periodic orbit (see figure 2 and text), so we have picked a higher threshold $\Psi_{thr}^2 = 1.55/V$, and the amplitude distribution largely deviates from a Gaussian.

Figure 5: The same as in figure 3 but for a strongly localized eigenfunction (state 8 on figs. 1 and 2) and the same threshold $\Psi_{thr}^2 = 1.55/V$ as in fig. 4. The state is strongly enhanced on the projection of a (2+2)D invariant chaotic submanifold \mathcal{R}_z , $z = 0$, so it is a beautiful example of a 2D scar. Correspondingly, the amplitude distribution deviates from a Gaussian.